FOURIER TRANSFORM, L^2 RESTRICTION THEOREM, AND SCALING

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Abstract. We show, using a Knapp-type homogeneity argument, that the (L^p, L^2) restriction theorem implies a growth condition on the hypersurface in question. We further use this result to show that the optimal (L^p, L^2) restriction theorem implies the sharp isotropic decay rate for the Fourier transform of the Lebesgue measure carried by compact convex finite hypersurfaces.

Section 0: Introduction

Let S be a smooth compact finite type hypersurface. Let $F_S(\xi) = \int_S e^{-i\langle x,\xi\rangle} d\sigma(x)$, where $d\sigma$ denotes the Lebesgue measure on S. Let $\mathcal{R}f = \hat{f}|_S$, where $\hat{f}(\xi)$ denotes the standard Fourier transform of f. It is well known (see [T], [Gr]) that if $|F_S(\xi)| \leq C(1+|\xi|)^{-r}$, $r > 0$, then $\mathcal{R} : L^p(\mathbb{R}^n) \to L^2(S)$ for $p \leq \frac{2(r+1)}{r+2}$. A natural question to ask is, does the boundedness of $\mathcal{R} : L^{\frac{2(r+1)}{r+2}}(\mathbb{R}^n) \to L^2(S)$, $r > 0$, imply that $|F_S(\xi)| \leq C(1+|\xi|)^{-r}$? In this paper we will show that this is indeed the case if S is a smooth convex finite type hypersurface in the sense that the order of contact with every tangent line is finite. (See [BNW]).

Under a more general condition, called the finite polyhedral type assumption, (see Definition 1 below), we will show that the (L^p, L^2) restriction theorem with $p = \frac{2(r+1)}{r+2}$ implies that $|B(x,\delta)| \leq C\delta^r$, where $B(x,\delta) = \{y : dist(y, T_x(S)) \leq \delta\}$, where $T_x(S)$ denotes the tangent hyperplane to S at x.

Our plan is as follows. We will first use a variant of the Knapp homogeneity argument to show that if S satisfies the finite polyhedral type condition and $\mathcal{R} : L^p(\mathbb{R}^n) \to L^2(S)$, with $p = \frac{2(r+1)}{r+2}$, then $|B(x,\delta)| \leq C\delta^r$ for each x. If the surface is, in addition, convex and finite type, then the result due to Bruna, Nagel, and Wainger (see [BNW]) implies that $|F_S(\xi)| \leq C(1+|\xi|)^{-r}$. If the surface is not convex finite type, then we do not, in general, know how to conclude that $|B(x,\delta)| \leq C\delta^r$ implies that $|F_S(\xi)| \leq C(1+|\xi|)^{-r}$. A gap remains.

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Section 1: Statement of Results

Definition 1. Let S be a smooth compact hypersurface in $\mathbb{R}^n$. Let $B^\pi(x,\delta)$ denote the projection of $B(x,\delta)$ onto $T_x(S)$. We say that S is of finite polyhedral type if there exists a family of polyhedra $P(x,\delta)$ such that $B^\pi(x,\delta) \subset P(x,\delta)$, $C_1|B^\pi(x,\delta)| \leq |P(x,\delta)| \leq C_2|B^\pi(x,\delta)|$, where C_1, C_2 do not depend on δ, and the number of vertices of $P(x,\delta)$ is bounded above independent of δ, where $\chi_{P(x,\delta)}$ denotes the characteristic function of $P(x,\delta)$.

Remark. The motivation for the definition of finite polyhedral type is the standard homogeneity argument due to Knapp. In order to prove the sharpness of the (L^p, L^2) restriction theorem for hypersurfaces with non-vanishing Gaussian curvature, Knapp approximated such a surface with a box with side-lengths $(\delta, \dots, \delta, \delta^2)$, δ small. He then took f_δ to be the inverse Fourier transform of the characteristic function of that box. It is not hard to see that $||\mathcal{R}f||_2 \approx \delta^{\frac{n-1}{2}}$. On the other hand, using the fact that the Fourier transform of the box in question is $\approx \frac{sin(\delta^2 x_n)}{x_n}\Pi_{j=1}^{n-1}\frac{sin(\delta x_j)}{x_j}$, it is not hard to see that $||f_\delta||_p \approx \delta^{\frac{n+1}{p'}}$. It follows that $p \leq \frac{2(n+1)}{n+3}$, which is the known positive result due to Stein and Tomas. The crucial part of this calculation is the approximation of the surface with a box with appropriate dimensions. Definition 1 and Theorem 2 are generalizations of this phenomenon.

It should also be noted that it is not hard to see that the hypersurface $S = \{x : x_3 = x_1x_2\}$ does not satisfy the finite polyhedral type condition. Thus, it makes sense to think of the finite polyhedral type condition as a generalization of convexity.

Theorem 2. *Let S be of finite polyhedral type. Consider the estimates*

$$|F_S(\xi)| \leq C(1+|\xi|)^{-r}, \tag{*}$$

$$\mathcal{R} : L^p(\mathbb{R}^n) \to L^2(S), \quad p \leq \frac{2(r+1)}{r+2}, \tag{**}$$

and

$$|B(x,\delta)| \leq C\delta^r, \tag{***}$$

for each x.

Then $(*)$ implies $(**)$ and $(**)$ implies $(***)$. Further, $(***)$ implies $(*)$ if S is in addition convex and finite type.

The fact that $(*)$ implies $(**)$ is essentially the Stein-Tomas restriction theorem. (See [T], [Gr]). The fact that $(***)$ implies $(*)$ in the case of convex finite type hypersurfaces is due to Bruna, Nagel, and Wainger. (See [BNW]). So it remains to prove that $(**)$ implies $(***)$, and that convex finite type hypersurfaces are of finite polyhedral type. (See Theorems 3 and 4 below).

Theorem 3. *Let* $S = \{x : x_n = Q(x') + R(x') + c\}$, *where* $x' = (x_1, \ldots, x_{n-1})$, $Q \in C^\infty$ *is mixed homogeneous in the sense that there exist integers* $(a_1, \ldots, a_{n-1})$, $a_j \geq 1$, *such that* $Q(t^{\frac{1}{a_1}} x_1, \ldots, t^{\frac{1}{a_{n-1}}} x_{n-1}) = tQ(x')$, $Q(x') \neq 0$ *if* $x' \neq (0, \ldots, 0)$, $R \in C^\infty$ *is the remainder in the sense that* $\lim_{t \to 0} \frac{R(t^{\frac{1}{a_1}} x_1, \ldots, t^{\frac{1}{a_{n-1}}} x_{n-1})}{t} = 0$, *and* c *is a constant. Then* S *is of finite polyhedral type.*

Theorem 3 implies that convex finite type hypersurfaces are of finite polyhedral type via the following representation result due to Schulz. (See [Sch]. See also [IS2]).

Theorem 4. *Let* $\Phi \in C^\infty(\mathbb{R}^{n-1})$ *be a convex finite type function such that* $\Phi(0, \ldots, 0) = 0$ *and* $\nabla\Phi(0, \ldots, 0) = (0, \ldots, 0)$. *Then, after perhaps applying a rotation, we can write* $\Phi(y) = Q(y) + R(y)$, *where* Q *is a convex polynomial, mixed homogeneous in the sense of Theorem 3, and* R *is the remainder in the sense of Theorem 3.*

Section 2: $(**) \to (***)$

Locally, S is a graph of a smooth function Φ, such that $\Phi(0, \ldots, 0) = 0$, and $\nabla\Phi(0, \ldots, 0) = (0, \ldots, 0)$. If we consider a sufficiently small piece of our hypersurface, $B^\pi(0, \ldots, 0, \delta) = \{y \in K : \Phi(y) \leq \delta\}$, where K is a compact set in $\mathbb{R}^{n-1}$ containing the origin, and, without loss of generality, $\Phi(y) \geq 0$. Since $|B^\pi(x, \delta)| \approx |B(x, \delta)|$, it suffices to show that $|\{y \in K : \Phi(y) \leq \delta\}| \leq C\delta^r$.

Let f_δ be a function such that $\hat{f}_\delta$ is the characteristic function of the set $\{(x', x_n) : x' \in P_\delta \ ; \ 0 \leq x_n \leq \delta\}$, where P_δ is the polyhedron containing the set $\{x' : \Phi(x') \leq \delta\}$ given by the definition of finite polyhedral type.

Let's assume for a moment that $||f_\delta||_p \leq C(\delta|P_\delta|)^{1/p'}$. Since the restriction theorem holds, we must have $||\mathcal{R}f_\delta||_2 \leq C||f_\delta||_p$, which implies that $|P_\delta| \leq C\delta^{\frac{2(p-1)}{2-p}}$. Since $p = \frac{2(r+1)}{r+2}$, it follows that $|P_\delta| \leq C\delta^r$. By the definition of P_δ it follows that $|B(0, \ldots, 0, \delta)| = |B^\pi(0, \ldots, 0, \delta)| = |\{y \in k : \Phi(y) \leq \delta\}| \leq C\delta^r$.

This completes the proof provided that we can show that $||f_\delta||_p \leq C(\delta|P_\delta|)^{1/p'}$. More generally, we will show that if P is a polyhedron in $\mathbb{R}^n$, then $||\widehat{\chi}_P||_p \leq C|P|^{1/p'}$, where C depends on the dimension and the number of vertices of P and $|P|$ denotes the volume of P. We give the argument in two dimensions, the argument in higher dimensions being similar. Break up P as a union of disjoint (up to the boundary) triangles t_j, $j = 1, \ldots, N$. Since $\chi_P(x) = \sum_j \chi_{t_j}(x)$, it suffices to carry out the argument for χ_P, where P is assumed to be a triangle. Since translations don't contribute anything in this context, we may assume that one of the vertices of the triangle is at the origin. Break up this triangle, if necessary, into two right triangles. Refine the original decomposition so that it consists of right triangles. Rotate the right triangle so that it is in the first quadrant and one of the sides is on the x_1-axis. We now apply a linear transformation mapping this triangle (denoted by P') into

the triangle with the endpoints $(0,0)$, $(1,0)$ and $(1,1)$. It is easy to check by an explicit computation that the Fourier transform of the characteristic function of this triangle has the L^p norm (crudely) bounded by $2^{\frac{1}{p}}$.

Let T denote the linear transformation taking the triangle P' to the unit triangle above. We see that

$$\widehat{\chi}_{TP'}(\xi) = |T|\widehat{\chi}_{P'}(T^t\xi),$$

so

$$\|\widehat{\chi}_{TP'}\|_p = |T|^{1/p'}\|\widehat{\chi}_{P'}\|_p.$$

Since $|T| = \frac{1}{|P'|}$, we see that $\|\widehat{\chi}_{t_j}\|_p \leq C|t_j|^{1/p'}$, where the t_j's are the triangles from the (refined) original decomposition. Adding up the estimates we get

$$\|\widehat{\chi}_P\|_p \leq C\sum_{j=0}^{N}|t_j|^{1/p'} \leq CN\Big(\sum_{j=0}^{N}|t_j|\Big)^{1/p'} = CN|P|^{1/p'}.$$

In higher dimensions the proof is virtually identical with triangles replaced by $n-1$ dimensional simplices, i.e the convex hull of n points that are not contained in any $(n-2)$ dimensional plane.

Since we have assumed that the number of vertices of P_δ is bounded above, it follows that $\|f_\delta\|_p \leq C(\delta|P_\delta|)^{1/p'}$, as desired.

Section 3: Proof of Theorem 3

As before, it is enough to consider the set $B^\pi(0,\dots,0) = \{y : Q(y)+R(y) \leq \delta\}$. It will be clear from the proof below that if we shrink the support sufficiently, then $B^\pi(0,\dots,0) \approx B_Q^\delta = \{y : Q(y) \leq \delta\}$, due to our assumptions on the remainder term R.

Let $\frac{n-1}{m} = \frac{1}{a_1} + \cdots + \frac{1}{a_{n-1}}$. Our plan is as follows. We first prove that $|B_Q^\delta| \approx \delta^{\frac{n-1}{m}}$. Then, we will find a polyhedron of suitable area that contains the set B_Q^1. We shall obtain the polyhedra for all values of δ by homogeneity.

Going into polar coordinates, $x_1 = s^{\frac{m}{a_1}}\omega_1, \dots, x_{n-1} = s^{\frac{m}{a_{n-1}}}$, $\omega = (\omega_1,\dots,\omega_{n-1}) \in S^{n-2}$, we see that $\int_{B_Q^\delta} dy = \int_{S^{n-2}}\int_0^{s^{\frac{1}{m}}Q^{-\frac{1}{m}}(\omega)} s^{n-2}\,ds\,d\omega = \delta^{\frac{n-1}{m}}\int_{S^{n-2}} Q^{-\frac{n-1}{m}}(\omega)\,d\omega = C_Q\delta^{\frac{n-1}{m}}$. This proves that $|B_Q^\delta| \approx \delta^{\frac{n-1}{m}}$.

We now find a box P_1 with sides parallel to the coordinate axes, such that $B_Q^1 \subset P_1$, and $|P_1| = cC_Q$, where $c > 1$. Let Q_P be a mixed homogeneous function of degree $(a_1,\dots,a_{n-1})$ defined by the condition $\{y : Q_P(y) = 1\} = \partial P_1$, where ∂P_1 denotes the boundary of P_1. Let P_δ be the polyhedron such that the boundary $\partial P_\delta = \{y : Q_P(y) = \delta\}$. It is not hard to see that $B_Q^\delta \subset P_\delta$. Moreover, $|P_\delta| = cC_Q\delta^{\frac{n-1}{m}} \approx |B_Q^\delta|$ by the calculation made in the previous paragraph. This completes the proof of Theorem 3.

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